Properties of schemes (see Har II3, Shaf V3.3–3.5)

Def: A scheme $(X, \mathcal{O}_X)$ is reduced if $\mathcal{O}_X(U)$ has no nilpotents for each open $U \subseteq X$, or equivalently (by a HW problem), if $\mathcal{O}_{X,P}$ has no nilpotents for each $P \in X$.

Each scheme has an associated reduced scheme w/ the same underlying topological space

$$X_{red} := (X, (\mathcal{O}_X)_{red})$$

where $(\mathcal{O}_X)_{red}$ is the sheafification of the presheaf

$$U \longmapsto \mathcal{O}_X(U)/\mathcal{N}$$

where $\mathcal{N}$ is the nilradical of $\mathcal{O}(U)$.

We can learn about the geometry of a scheme X near a point by looking at morphisms from a nonreduced point to X:

Let $X = \operatorname{Spec} R$, and $P \in X$. Then the natural map

$$R \longrightarrow k(P) = R_P/PR_P$$

corresponds to the inclusion of the point $\operatorname{Spec} k(P) \hookrightarrow X$ with image P.

If R is a k-algebra, then $P \in X$ is called a <u>k-point</u> if $k(P) = k$. Then a morphism of schemes

$$\operatorname{Spec} k \longrightarrow \operatorname{Spec} R$$

is equivalent to the choice of a (closed) k-point.

Now, set $D = k[\varepsilon]/(\varepsilon^2)$, which has k-basis $1, \varepsilon$.

Then a map $\operatorname{Spec} D \to \operatorname{Spec} R$ corresponds to

$$\varphi : R \longrightarrow k[\varepsilon]/(\varepsilon^2).$$

Let $m = \varphi^{-1}((\varepsilon))$. Then m is the k-point in the image of $\operatorname{Spec} D$, and φ factors as

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & & k[\varepsilon]/(\varepsilon^2) \cong k \oplus \varepsilon k \\ & R/m^2 \arrow[ur] & \end{tikzcd}$$

The restriction $m/m^2 \longrightarrow k \oplus \varepsilon k$ then has image $\varepsilon k \cong k$, which thus gives an element of $\operatorname{Hom}_k(m/m^2, k)$, i.e. a tangent vector!

Conversely, given a k-point p and a tangent vector $\alpha \in \operatorname{Hom}_k(m_p/m_p^2, k)$, define a homomorphism

$$R/m_p^2 \longrightarrow k[\varepsilon]/(\varepsilon^2)$$

such that $f \longmapsto \underbrace{f(p)}_{\substack{\text{image of} \\ f \text{ in quotient} \\ R/m_p = k}} + \alpha\big(f - f(p)\big)\varepsilon.$ (Check that this is a homomorphism)

Then take $R \longrightarrow k[\varepsilon]/(\varepsilon^2)$ to be the composition with the quotient.

Thus, a map of schemes $\operatorname{Spec}\left(k[\varepsilon]/(\varepsilon^2)\right) \longrightarrow \operatorname{Spec} R$ is the same as a choice of a k-point of $\operatorname{Spec} R$ together with a tangent vector at that point.

More generally, let R be any ring and $m \in \operatorname{Spec} R$ a closed point. Then we get a sequence of subschemes

$$\operatorname{Spec} R/m \subseteq \operatorname{Spec} R/m^2 \subseteq \operatorname{Spec} R/m^3 \subseteq \dots \subseteq \operatorname{Spec} R$$

which is each just a point w/ more and more extra information.

<u>Ex</u>: Let $R = k[x,y]$, $m = (x,y)$. Then the map

$$R \to R/m^2 \quad \text{sends}$$

$$f \longmapsto f(0,0) + \underbrace{f_x(0,0)}_{\text{partial derivative}}\, x + f_y(0,0)\, y$$

So $\operatorname{Spec} R/m^2$ retains the value of a function at

m along with the values of its first derivatives.

Spec R/m^3 retains the value of its second derivs as well, etc.

Side note: The inverse limit of these rings is the completion of R at m:

$$\hat{R} := \varprojlim R/m^i \subseteq \prod_i R/m^i$$

Spec $\hat{R}$ isn't a subscheme of Spec R, but it's called an "analytic neighborhood" of m since it retains the info of all the partial derivatives.

Def: A scheme X is irreducible if its topological space is irreducible. i.e. it can't be written as the union of two proper closed sets.

Def: X is integral if for every open $U \subseteq X$, $\mathcal{O}_X(U)$ is an integral domain.

Ex: For $X = \operatorname{Spec} R$ an affine scheme:

- X is irreducible $\Leftrightarrow$ $\mathfrak{N} = \sqrt{(0)}$ is prime
- X is reduced $\Leftrightarrow$ $\mathfrak{N} = 0$ (Do you see why?)
- X is integral $\Leftrightarrow$ R is an integral domain

Prop: A scheme X is integral $\iff$ it's reduced and irreducible.

Pf: $(\Rightarrow)$ Assume X is integral. Clearly it's reduced.

If it's reducible as $X = X_1 \cup X_2$, take $U_1 = X \setminus X_1$, $U_2 = X \setminus X_2$, two disjoint open sets. Then $\mathcal{O}(U_1 \cup U_2) = \mathcal{O}(U_1) \times \mathcal{O}(U_2)$, which isn't an integral domain, since $(1,0)\cdot(0,1) = 0$. Thus, X is irreducible.

$(\Leftarrow)$ Now assume reduced and irreducible. Let $U \subseteq X$ be open and assume $f, g \in \mathcal{O}(U)$ st. $fg = 0$.

Let $Y = \{x \in U \mid f_x \in m_x\}$, $Z = \{x \in U \mid g_x \in m_x\}$

germ in stalk

For any affine open $V = \operatorname{Spec} A \subseteq U$, we have $(f|_V)_x \in m_x \iff f|_V \in x \iff x \in V(f|_V)$. Thus, $Y \cap V = V(f)$ and $Z \cap V = V(g)$. So Z and Y are closed.

Moreover, at each $x \in U$, $f_x g_x = 0 \in m_x$, so f_x or $g_x \in m_x$. Thus $Y \cup Z = U$.

But X is irreducible, so U is as well. (Do you see why?)

Thus, WLOG, $Y = U$. But then in A, f is in every prime ideal, so f is nilpotent, so $f = 0$. Thus, X is integral.

Def: A scheme X is <u>locally Noetherian</u> if it can be covered by affine opens $\operatorname{Spec} A_i$, where each A_i is Noetherian. (Equivalently, for every affine open $U = \operatorname{Spec} A$, A is Noetherian. — Details in Hartshorne.)

If X is also quasicompact, then it's <u>Noetherian</u>. (Equivalently X Noetherian $\Leftrightarrow$ it can be covered by finitely many open affines $\operatorname{Spec} A_i$ w/ each A_i Noetherian)

In particular, $\operatorname{Spec} A$ is Noetherian $\Leftrightarrow$ A is Noetherian.

Def: A morphism $f: X \to Y$ of schemes is <u>locally of finite type</u> if for every affine open $\operatorname{Spec} B \subseteq Y$, $f^{-1}(\operatorname{Spec} B)$ can be covered by affine opens $\operatorname{Spec} A_i$ s.t. each A_i is a f.g. B-algebra.
f is of <u>finite type</u> if we can choose only finitely many $\operatorname{Spec} A_i$.

Note: Most "geometric" examples of morphisms will be of finite type. We have the following:

Prop: If $\varphi : B \to A$ is a ring homomorphism, then A is a f.g. B-algebra $\iff$ $\operatorname{Spec} A \to \operatorname{Spec} B$ is of finite type. (Exercise — see Shaf)

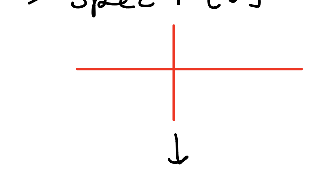
Def: A morphism $f : X \to Y$ is <u>finite</u> if for every affine open $\operatorname{Spec} B \subseteq Y$, $f^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$ for some A and A is a f.g. B-module. (Note: this is much stronger than f.g. B-algebra.)

Remark: For (locally) finite type and finite, we can instead replace all open affines in Y with those in an open cover.

Claim: Finiteness is transitive: i.e. the composition of finite morphisms is finite. This is because finite generation of modules is transitive.

Ex: For any surjective ring map $B \twoheadrightarrow A$, $\operatorname{Spec} A \to \operatorname{Spec} B$ is finite.

Ex: Consider the map $\operatorname{Spec} k[x,y]/(xy) \to \operatorname{Spec} k[t]$ given by the ring map $t \mapsto x$.

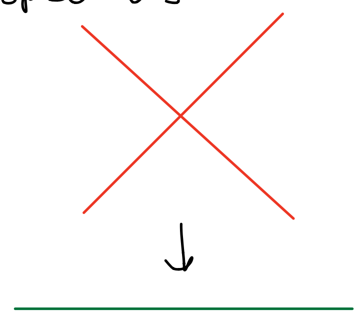


This isn't finite!

However, $\operatorname{Spec} k[x,y]/(x^2-y^2) \to \operatorname{Spec} k[t]$

via $t \longmapsto x$ is:

$$k[x,y]/(x^2-y^2) = k[t]1 + k[t]x$$



Ex: If $X \to \operatorname{Spec} k$ is finite, then X is finite. (HW)

Idea: X must be $\operatorname{Spec} A$, so A is a finite dim k-vector space.

Claim: For $P \in \operatorname{Spec} A$, A/p is a field.

Thus, all ideals are maximal. Apply Chinese remainder Theorem and bound # of max'l ideals.
(Alternately, can show A is Artinian, which implies $\operatorname{Spec} A$ is finite — requires more CA. We'll prove this in 523.)

Def: An <u>open subscheme</u> of X is a scheme U s.t. U is open in X, and $\mathcal{O}_U \cong \mathcal{O}_X|_U$. An <u>open immersion</u> is a morphism $f: X \to Y$ which induces an isomorphism $(X, \mathcal{O}_X) \to (f(X), \mathcal{O}_Y|_{f(X)})$ where $f(X) \subseteq Y$ is open.

Ex: Consider the open immersion
$$\mathbb{A}^2_k - \{(x,y)\} \longrightarrow \mathbb{A}^2_k.$$

This is not finite since $\mathbb{A}^2_k$ minus the origin is not affine! (Otherwise, since the induced map on global sections is an isomorphism, the immersion itself must be an isomorphism, which it's not.)

However, it is of finite type: take the covering

$$\begin{array}{ccc} D(x) & \cup\ D(y) & = \mathbb{A}^2_k - \{(x,y)\} \\ \| & \| & \\ \operatorname{Spec} k[x,y]_x & \operatorname{Spec} k[x,y]_y & \\ \| & \| & \\ k[x,y][1/x] & k[x,y][1/y] & \end{array}$$

f.g. $k[x,y]$-algebras

Def: A <u>closed immersion</u> is a morphism $f: Y \to X$ such that the image is closed, f induces a homeomorphism of Y onto its image, and the induced map $f^{\#}: \mathcal{O}_X \to f_* \mathcal{O}_Y$ is surjective. A <u>closed subscheme</u> of X is an equivalence class of closed immersions, where $f: Y \to X$ is equivalent to $f': Y' \to X$ if there's an isomorphism $i: Y \to Y'$ s.t. $f' \circ i = f$

$$\begin{tikzcd} Y \arrow[rr, "i", "\cong"'] \arrow[dr, "f"'] & & Y' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

The notion of closed subscheme is subtle!

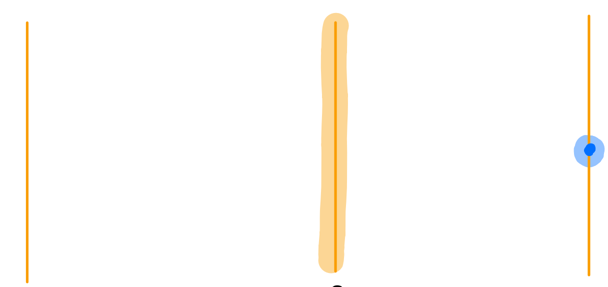
Ex: Let $R = k[x,y]$. Then $R \twoheadrightarrow R/(x)$ corresponds to the closed immersion of the y-axis into $\mathbb{A}^2$.

However, take $R \twoheadrightarrow R/(x^2)$. The set-theoretic image of the Spec map is the same, but it gives a non-reduced structure on the y-axis.

$R \to R/(x^2, xy)$ corresponds to a different closed subsch. structure on the y-axis! since

$$\left(R/(x^2, xy)\right)_{(x, y-a)} \quad \text{has nilpotents} \iff a = 0$$

this has "embedded component" (x,y).



$R \to R/(x)$ — reduced structure on y-axis

$R \to R/(x^2)$ — non-reduced everywhere

$R \to R/(x^2, xy)$ — nilpotents at origin

Note: $k((x,y)) = k$ in each of these schemes, but the dimension of the tangent space at (x,y) is one in the first example and 2 in the others.

In general, if $Y \subseteq X$ is closed, there will be a unique "smallest" closed subscheme structure on Y, called the <u>reduced induced closed subscheme structure</u> on Y.

If $X = \operatorname{Spec} A$, and $Y = V(I)$, it is given by $\operatorname{Spec} A/\sqrt{I} \to \operatorname{Spec} A$. For an arbitrary scheme X, it is given by glueing the subschemes obtained from $X_i \cap Y$ for X_i open affines.

Def: The <u>dimension</u> of a scheme X is the supremum of lengths of chains of closed irreducible subsets:

$$\emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n$$

If $X = \operatorname{Spec} R$, then $\dim X = \dim R$.

If $Z \subseteq X$ is an irreducible closed subset, then the <u>codimension</u> of Z in X is the supremum of lengths of chains of closed irreducible subsets

$$Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

If $Y \subseteq X$ is any closed subset of X (possibly reducible) then $\operatorname{codim}(Y, X) = \inf\limits_{\substack{Z \subseteq Y \\ \text{closed,} \\ \text{irreducible}}} \operatorname{codim}(Z, X)$